Low-lying spectrum of the Y-string three-quark potential using hyper-spherical coordinates

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We calculate the energies of three-quark states with definite permutation symmetry (i.e. of SU(6) multiplets) in the N=0,1,2 shells, confined by the Y-string three-quark potential. The exact Y-string potential consists of one, so-called three-string term, and three angle-dependent two-string terms. Due to this technical complication we treat the problem at three increasingly accurate levels of approximation: 1) the (approximate) three-string potential expanded to first order in trigonometric functions of hyper-spherical angles; 2) the (approximate) three-string potential to all orders in the power expansion in hyper-spherical harmonics, but without taking into account the transition(s) to two-string potentials; 3) the exact minimal-length string potential to all orders in power expansion in hyper-spherical harmonics, and taking into account the transition(s) to two-string potentials. We show the general trend of improvement of these approximations: The exact non-perturbative corrections to the total energy are of the order of one per cent, as compared with approximation 2), yet the exact energy differences between the $[20, 1^+], [70, 2^+], [56, 2^+], [70, 0^+]$-plets are shifted to 2:2:0.9, from the Bowler and Tynemouth separation rule 2:2:1, which is obeyed by approximation 2) at the one per cent level. The precise value of the energy separation of the first radial excitation ("Roper") $[56', 0^+]$-plet from the $[70, 1^-]$-plet depends on the approximation, but does not become negative, i.e. the "Roper" remains heavier than the odd-parity $[70, 1^-]$-plet in all of our approximations.

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I. INTRODUCTION

The so-called Y-junction string three-quark potential, defined by

$$V_Y = \sigma \min_{\mathbf{x_0}} \sum_{i=1}^{3} |\mathbf{x_i} - \mathbf{x_0}|. \tag{1}$$

has long been advertised [1, 2, 3, 4, 5, 6] as the natural approximation to the flux tube confinement mechanism, that is allegedly active in QCD. Lattice investigations, Refs. [7, 8, 9], however, contradict each other in their attempts to distinguish between the Y-string, Fig. 1, and the Δ-string potential, see Fig. 2,

$$V_\Delta = \sigma \sum_{i<j=1}^{3} |\mathbf{x}_i - \mathbf{x}_j|, \tag{2}$$

which, in turn, is indistinguishable from the sum of three linear two-body potentials. The present point of view held among the lattice QCD practitioners is that there should be a smooth cross-over from the Δ to the Y-potential at interquark distances of around 0.8 fm [10]. Exactly how this cross-over should be implemented is not clear just now. Moreover, quantum fluctuations of the three flux tubes lead to Lüscher type corrections in the potential [11]. The evaluation of such corrections is beyond the scope of the present paper.

Over the past 25 years, the Y-string potential has been used in several, more or less systematic studies of baryons in the (constituent) quark model with various hyperfine interactions [3, 4, 5, 12], and yet some of the most basic predictions of this potential acting alone, i.e. without hyperfine interaction, such as the spectrum of the low-lying three-quark states remain unknown. The older calculations [3, 4] treat the Y-string potential only in first-order perturbation theory, whereas the more recent ones [12] invoke equivalence with Δ string potential[26], up to an

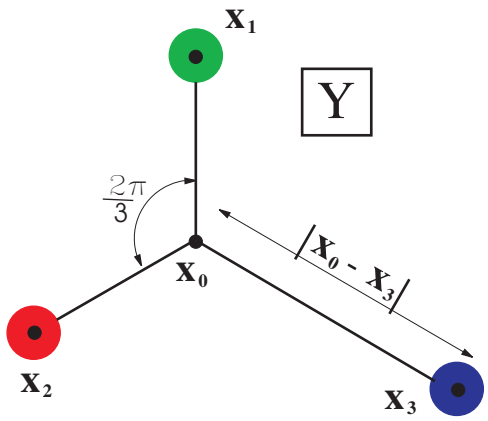

FIG. 1: Three-quark Y-junction string potential.

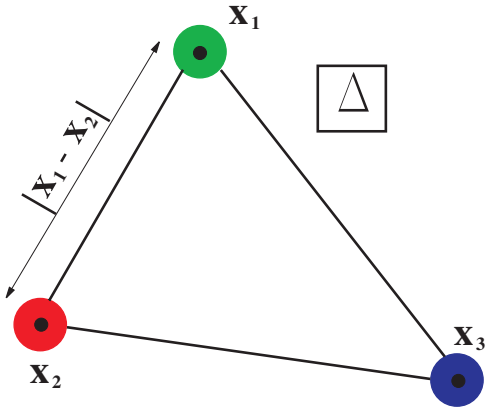

FIG. 2: Three-quark Δ-shape string potential.

overall multiplicative factor $f \simeq 0.5493$ ($\frac{1}{2} < f < \frac{1}{\sqrt{3}}$) for the string tension σ. That leads us to a third set of calculations that rely on the (as yet not proven) "equivalence" of the Y-string potential and the Δ string one to look for the "best two-body approximants" to the Y-string potential, see Refs. [13, 22] and follow-up references. Note, however, that these papers use only the hyper-radial approximation and they are only concerned with the overall strength of the coupling.

We wish to resolve this dilemma using analytical methods, whenever possible [27]. So, at least for the time being, we have to resort to calculations in the nonrelativistic quark model. We shall not repeat the two-body Δ string potential calculation, as there is a large body of literature on the subject: instead, we shall use the paper by J.M. Richard and P. Taxil [19] as a benchmark. That paper uses the hyper-spherical harmonics method, which we also use, with only a slightly different practical implementation: whereas Ref. [19] uses the Jacobi coordinates (see Sect. II) in the evaluation of the relevant matrix elements, we shall evaluate them directly in the hyperspherical variables.

For the sake of completeness and clarity we shall start with two extensive technical preparatory sections: First, in Sect. II we outline the three main technical problems, and then in Sect. III we address them one after another; for this reason we divide Sect. III, in which we formulate the methods that we use, into three parts: the first one, Sect. III A, is about the angular momentum recoupling algebra necessary to deal with the non-conserved "partial" angular momenta, the second part, Sect. III B, deals with the square root(s) in the Y-string potential, and finally the third one, Sect. III C, addresses all four forms of the string potential together. Our results are shown in Sect. IV, which is divided into four parts: one part that comprises Sect. IV A deals with the first-order perturbative approximation results, wherein the angular dependence is treated as a (small) perturbation to the linear potential oscillator, Sect. IV A 1 and two with non-perturbative ones in Sect. IV B: one for the Y-string potential in Sect. IV B 1, and another for the complete string potential in Sect. IV B 2. The final Section V contains a summary and discussion of our results. A number of technical issues are discussed in Appendices A,B,C.

II. THE Y-STRING POTENTIAL

The complexity of the potential Eq. (1) is best seen when expressed in terms of three-body Jacobi (relative) coordinates $\boldsymbol{\rho}, \boldsymbol{\lambda}$

$$\boldsymbol{\rho} = \frac{1}{\sqrt{2}}(\mathbf{x_1} - \mathbf{x_2}), \tag{3}$$

$$\boldsymbol{\lambda} = \frac{1}{\sqrt{6}}(\mathbf{x_1} + \mathbf{x_2} - 2\mathbf{x_3}), \tag{4}$$

as follows. The exact string potential Eq. (1) consists of the so-called Y-string, or three-string term,

$$V_{\text{string}} = V_{\text{Y}} = \sigma\sqrt{\frac{3}{2}(\boldsymbol{\rho}^2 + \boldsymbol{\lambda}^2 + 2|\boldsymbol{\rho} \times \boldsymbol{\lambda}|)}, \tag{5}$$

$$\text{when} \begin{cases} 2\boldsymbol{\rho}^2 - \sqrt{3}\boldsymbol{\rho}\cdot\boldsymbol{\lambda} \geq -\rho\sqrt{\boldsymbol{\rho}^2 + 3\boldsymbol{\lambda}^2 - 2\sqrt{3}\boldsymbol{\rho}\cdot\boldsymbol{\lambda}} \\ 2\boldsymbol{\rho}^2 + \sqrt{3}\boldsymbol{\rho}\cdot\boldsymbol{\lambda} \geq -\rho\sqrt{\boldsymbol{\rho}^2 + 3\boldsymbol{\lambda}^2 + 2\sqrt{3}\boldsymbol{\rho}\cdot\boldsymbol{\lambda}} \\ 3\boldsymbol{\lambda}^2 - \boldsymbol{\rho}^2 \geq -\frac{1}{2}\sqrt{(\boldsymbol{\rho}^2 + 3\boldsymbol{\lambda}^2)^2 - 12(\boldsymbol{\rho}\cdot\boldsymbol{\lambda})^2}\,. \end{cases}$$

and three angle-dependent two-part string, or the so-called V-string, terms,

$$V_{\text{string}} = \sigma\left(\sqrt{\frac{1}{2}(\boldsymbol{\rho}^2 + 3\boldsymbol{\lambda}^2 + 2\sqrt{3}\boldsymbol{\rho}\cdot\boldsymbol{\lambda})} + \sqrt{\frac{1}{2}(\boldsymbol{\rho}^2 + 3\boldsymbol{\lambda}^2 - 2\sqrt{3}\boldsymbol{\rho}\cdot\boldsymbol{\lambda})}\right) \tag{6a}$$

$$\text{when} \begin{cases} 2\boldsymbol{\rho}^2 - \sqrt{3}\boldsymbol{\rho}\cdot\boldsymbol{\lambda} \geq -\rho\sqrt{\boldsymbol{\rho}^2 + 3\boldsymbol{\lambda}^2 - 2\sqrt{3}\boldsymbol{\rho}\cdot\boldsymbol{\lambda}} \\ 2\boldsymbol{\rho}^2 + \sqrt{3}\boldsymbol{\rho}\cdot\boldsymbol{\lambda} \geq -\rho\sqrt{\boldsymbol{\rho}^2 + 3\boldsymbol{\lambda}^2 + 2\sqrt{3}\boldsymbol{\rho}\cdot\boldsymbol{\lambda}} \\ 3\boldsymbol{\lambda}^2 - \boldsymbol{\rho}^2 \leq -\frac{1}{2}\sqrt{(\boldsymbol{\rho}^2 + 3\boldsymbol{\lambda}^2)^2 - 12(\boldsymbol{\rho}\cdot\boldsymbol{\lambda})^2} \end{cases}$$

$$V_{\text{string}} = \sigma\left(\sqrt{2}\rho + \sqrt{\frac{1}{2}(\boldsymbol{\rho}^2 + 3\boldsymbol{\lambda}^2 + 2\sqrt{3}\boldsymbol{\rho}\cdot\boldsymbol{\lambda})}\right) \tag{6b}$$

$$\text{when} \begin{cases} 2\boldsymbol{\rho}^2 - \sqrt{3}\boldsymbol{\rho}\cdot\boldsymbol{\lambda} \geq -\rho\sqrt{\boldsymbol{\rho}^2 + 3\boldsymbol{\lambda}^2 - 2\sqrt{3}\boldsymbol{\rho}\cdot\boldsymbol{\lambda}} \\ 2\boldsymbol{\rho}^2 + \sqrt{3}\boldsymbol{\rho}\cdot\boldsymbol{\lambda} \leq -\rho\sqrt{\boldsymbol{\rho}^2 + 3\boldsymbol{\lambda}^2 + 2\sqrt{3}\boldsymbol{\rho}\cdot\boldsymbol{\lambda}} \\ 3\boldsymbol{\lambda}^2 - \boldsymbol{\rho}^2 \geq -\frac{1}{2}\sqrt{(\boldsymbol{\rho}^2 + 3\boldsymbol{\lambda}^2)^2 - 12(\boldsymbol{\rho}\cdot\boldsymbol{\lambda})^2} \end{cases}$$

$$V_{\text{string}} = \sigma\left(\sqrt{2}\rho + \sqrt{\frac{1}{2}(\boldsymbol{\rho}^2 + 3\boldsymbol{\lambda}^2 - 2\sqrt{3}\boldsymbol{\rho}\cdot\boldsymbol{\lambda})}\right) \tag{6c}$$

$$\text{when} \begin{cases} 2\boldsymbol{\rho}^2 - \sqrt{3}\boldsymbol{\rho}\cdot\boldsymbol{\lambda} \leq -\rho\sqrt{\boldsymbol{\rho}^2 + 3\boldsymbol{\lambda}^2 - 2\sqrt{3}\boldsymbol{\rho}\cdot\boldsymbol{\lambda}} \\ 2\boldsymbol{\rho}^2 + \sqrt{3}\boldsymbol{\rho}\cdot\boldsymbol{\lambda} \geq -\rho\sqrt{\boldsymbol{\rho}^2 + 3\boldsymbol{\lambda}^2 + 2\sqrt{3}\boldsymbol{\rho}\cdot\boldsymbol{\lambda}} \\ 3\boldsymbol{\lambda}^2 - \boldsymbol{\rho}^2 \geq -\frac{1}{2}\sqrt{(\boldsymbol{\rho}^2 + 3\boldsymbol{\lambda}^2)^2 - 12(\boldsymbol{\rho}\cdot\boldsymbol{\lambda})^2} \end{cases}.$$

Here, the reasons for the lack of use of the exact potential Eq. (1) become clear: i) it is a genuine three-body operator with a complicated and unusual ("area term") angular dependence under the square-root of the most important term (the Y-junction string potential) that leads to the non-conservation of the individual Jacobi coordinates' angular momenta [14] and hugely complicates the equations of motion; ii) the square-roots appearing in all four functional forms of the potential make this task even more difficult; iii) the presence of four different functional forms of the potential depending on the configuration space angles makes the integration of the equations of motion difficult as one cannot easily separate the angular and radial integrals.

Perhaps the simplest, yet realistic approximation to the exact string potential Eq. (1) is the Y-string, or the "three-string" potential, Eq. (5), that is used in the whole configuration space, i.e., even when one of the angles in the triangle exceeds 120°. In that way one avoids the cumbersome transition to the V-string potentials, see problem iii) above [28]. Still, even this simplified approximation suffers from two difficulties mentioned above: i) an unusual ("area term") angular dependence under the square-root that leads to the non-conservation of the individual Jacobi coordinates' angular momenta; ii) the square-root. We shall address these problems in successive steps: i) the area term turns out to be exactly (analytically) integrable, but it requires complicated angular momenta recoupling algebra and the exact value of a particular one dimensional angular integral. Problem ii), the square root, can be treated, at first, by a series expansion, i.e. in perturbation theory, and then by numerical evaluation of the complete functional

expression, i.e. in non-perturbative approximation. Finally, the last issue iii) is tackled, at the price of considerable inconvenience: the necessary hyperangular matrix elements of the complete three-body potential can be evaluated using above mentioned methods, except that two previously separate integrals (over one quasi-radial and one angular variable) have to be performed simultaneously, as the integration boundaries involve both variables.

It turns out that the crucial ingredient for the success of this effort is the application of the so-called hyper-spherical coordinates/angles [17], or a particular variation thereof, the cosines of the relative angle θ between the Jacobi coordinates $\boldsymbol{\rho}, \boldsymbol{\lambda}$ and of the angle 2χ defined by way of the ratio of the moduli ρ, λ of the two Jacobi coordinates.

The hyper-spherical method has been used widely in the few-body atomic and nuclear physics, for review see Ref. [24], but we are aware of only one paper, Ref. [22], that uses it in the context of the three-quark Y-string problem, which is perhaps ironic, as it turns out to be the most natural set of coordinates for the problem at hand, whereas its use in two-body atomic and nuclear potential problems is plagued by slow convergence of the expansion. Some papers have applied the hyper-radial approximation to the three-quark problem [13, 19], but that misses the essential points discussed here.

III. METHODS

The first problem (angular momentum) is generic to all (string) three-body potentials and is thus independent of the approximation used, so its solution will be used subsequently in both perturbative and non-perturbative approximations. Moreover, its solution can be incorporated into the hyper-spherical formalism that will used later. For this reason we start with the angular momentum recoupling.

A. Angular momentum recoupling

The 'vector cross-product, or the "area term" $2\,|\boldsymbol{\rho}\times\boldsymbol{\lambda}\,|$ in this potential has some curious properties in classical and quantum mechanics: it conserves the sum $\mathbf{L} = \mathbf{l}_\rho + \mathbf{l}_\rho$ of the two partial (orbital) angular momenta $\mathbf{l}_\rho = \boldsymbol{\rho}\times\mathbf{p}_\rho$ and $\mathbf{l}_\lambda = \boldsymbol{\lambda}\times\mathbf{p}_\lambda$, but not their difference [14], i.e., the individual (orbital) angular momenta are not conserved. As a consequence, there can be "spilling" of the orbital angular momentum from the ρ (normal) mode into the λ one and *vice versa*. Only the radial excitations of the S-wave ground state remain immune to this spillage.

We need to evaluate matrix elements of the following form

$$\begin{aligned}\langle (l_{1f}\otimes l_{2f})\,L_f, M_f||\sin\theta||\,(l_{1i}\otimes l_{2i})\,L_i, M_i\rangle \;&=\; \sum_{m_{1f},m_{2f},m_{1i},m_{2i}} (l_{1f},m_{1f};l_{2f},m_{2f}|L_f,M_f)\\ (l_{1i},m_{1i};l_{2i},m_{2i}|L_i,M_i)\;&\times\; \langle l_{1f},m_{1f};l_{2f},m_{2f}||\sin\theta||l_{1i},m_{1i};l_{2i},m_{2i}\rangle \end{aligned} \tag{7}$$

where

$$|\,\hat\rho\times\hat\lambda\,| \;=\; \frac{|\,\boldsymbol{\rho}\times\boldsymbol{\lambda}\,|}{\rho\lambda} = |\sin\theta| = \sqrt{1-\cos^2\theta} \tag{8}$$

and

$$\begin{aligned}\cos\theta \;&=\; \cos\theta_1\,\cos\theta_2 + \cos(\varphi_1-\varphi_2)\,\sin\theta_1\,\sin\theta_2\\ &=\; -\left(\frac{4\,\pi}{\sqrt{3}}\right)\,[Y_1(\theta_1,\phi_1)\otimes Y_1(\theta_2,\phi_2)]_{L=0}\end{aligned} \tag{9}$$

that is to be inserted into

$$\begin{aligned}&\langle l_{1f},m_{1f},l_{2f},m_{2f}||\sin\theta||(l_{1i},m_{1i},l_{2i},m_{2i})\rangle \;=\\ \int d\Omega_1\int d\Omega_2 Y^*_{l_{1f},m_{1f}}(\theta_1,\phi_1)Y_{l_{1i},m_{1i}}(\theta_1,\phi_1)Y^*_{l_{2f},m_{2f}}(\theta_2,\phi_2)Y_{l_{2i},m_{2i}}(\theta_2,\phi_2)\;&\times\;|\sin\theta|\end{aligned} \tag{10}$$

Capstick and Isgur, Ref. [4] have reduced the angular matrix elements Eq. (7) to

$$\begin{aligned}\langle (l_{1f}\otimes l_{2f})L_f, M_f||\sin\theta||(l_{1i}\otimes l_{2i})L_i, M_i\rangle \;&=\; \delta_{L_f,L_i}\delta_{M_f,M_i}\sum_L\sqrt{(2l_{1f}+1)(2l_{2f}+1)(2l_{1i}+1)(2l_{2i}+1)}\\ &\times\; (-1)^{L+l_{1f}+l_{2f}}W(l_{1i},L,L_i,l_{2f};l_{1f},l_{2i})(l_{1f},0,l_{1i},0|L,0)(l_{2f},0,l_{2i},0|L,0)\\ &\times\; \frac{1}{2}\int_{-1}^{1} d\cos\theta P_L(\cos\theta)|\sin\theta|\end{aligned} \tag{11}$$

a sum of products of SU(2) Clebsch-Gordan $(l_{1f},0,l_{1i},0|L,0)(l_{2f},0,l_{2i},0|L,0)$ and Racah $W(l_{1i},L,L_i,l_{2f};l_{1f},l_{2i})$ coefficients, which can be found in standard collections of angular momentum tables, such as Ref. [20], and an integral over even-L order Legendre polynomials $P_L(x)$ that can be reduced to a ratio of double-factorial functions [23]:

$$\frac{1}{2}\int_{-1}^{1} d\cos\theta P_L(\cos\theta)\sqrt{1-\cos^2\theta} \;=\; \begin{cases} 0\ , & \text{for odd } L=1,3,5,... \\ \frac{\pi}{4}, & \text{for } L=0 \\ -\frac{\pi}{2L}\frac{(L-1)!!(L-3)!!}{(L+2)!!(L-2)!!}\ , & \text{for even } L=2,4,6,...\neq 0 \end{cases} \tag{12}$$

In Table I we show the dependence of these matrix elements on the "partial" orbital angular momenta l_ρ, l_λ - note in particular the non-vanishing off-diagonal matrix element that ensures the existence of the mass-splitting "mixing" terms in the $[56,2^+]$ and the $[70,2^+]$-plets. Note that all of the angular matrix elements are exact integer or fractional multiples of the basic unit $(\frac{\pi}{4})$, thus leading to the analytic form of the angular matrix elements, shown in the eighth column of Table I.

TABLE I: Non-vanishing diagonal and off-diagonal matrix elements $\langle|\sin\theta|\rangle = \langle(l_{1f}\otimes l_{2f})^{M_f}_{L_f}\,||\sin\theta||\,(l_{1i}\otimes l_{2i})^{M_i}_{L_i}\rangle$, in total S-, P- and D-wave states, and partial waves $l_{1f}=l_{2f}=l_{1i}=l_{2i}=0,1,2$.

N	L_f	$l_{\rho f}\times l_{\lambda f}$	L_i	$l_{\rho i}\times l_{\lambda i}$	$\frac{4}{\pi}\langle\|\sin\theta\|\rangle$
0	0	0×0	0	0×0	1
2	0	1×1	0	1×1	$\frac{3}{4}$
1	1	1×0	1	1×0	1
2	1	1×1	1	1×1	$\frac{9}{8}$
2	2	1×1	2	1×1	$\frac{39}{40}$
2	2	2×0	2	2×0	1
2	2	2×0	2	0×2	-$\frac{1}{8}$

B. Perturbative approximation

We must first determine the qualitative features of the "area term" $2\,|\boldsymbol{\rho}\times\boldsymbol{\lambda}\,|$ and its angular momentum dependence on the ordering of states in quantum mechanics, e.g. whether the S-wave states, such as the Roper, are lowered or raised in energy as compared with other states in the N=2 band, e.g. the P- and D-waves?

A historical remark seems in order now: It has been known at least since the late 1970's, see Refs. [15, 16], that arbitrary an-harmonic (both two- and three-body) potentials split the N=2 shell harmonic oscillator states[29] according to their spatial permutation symmetry classes, or, what is the same, according to the SU(6) multiplets. This splitting of the harmonic oscillator spectrum has been worked out to various degrees of mathematical sophistication in Refs. [15, 16, 18, 19] for arbitrary two-body potentials, but has been only briefly mentioned in the case of three-body ones [18]. In particular Bowler and Tynemouth [18] have shown, on the basis of the Sp(12,R) group theory and first-order perturbation theory, that the energy ordering and splitting of four of the five SU(6) multiplets in the N=2 band always remain the same for arbitrary permutation-symmetric three-body potentials, the only exception being the energy of the $[56,0^+]$ multiplet (containing the Roper resonance), which remains unconstrained by this theorem. In other words, the key to the mystery of the Roper's abnormally low mass may well reside in the form of the three-quark potential.

1. The square root in the Y-string potential

Perhaps the simplest way to address these questions is to expand the root in the Y-string potential Eq. (5) in a power series

$$V_{\rm Y} \;=\; \sigma\sqrt{\frac{3}{2}(\boldsymbol{\rho}^2+\boldsymbol{\lambda}^2)}\left(1+\frac{|\boldsymbol{\rho}\times\boldsymbol{\lambda}|}{\boldsymbol{\rho}^2+\boldsymbol{\lambda}^2}+\cdots\right). \tag{13}$$

and then, keep only a few lowest-order non-trivial terms, apply the first-order perturbation theory. The unperturbed potential in Eq. (13) is slightly more complicated than the harmonic oscillator, so we break this into two steps:

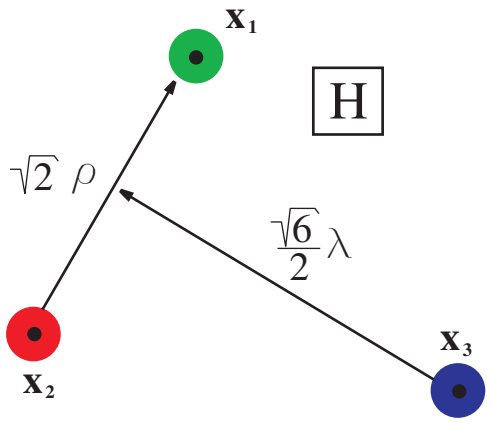

FIG. 3: Two three-body Jacobi coordinates ρ, λ that define the "hyper-radial" string length $\sqrt{|\rho|^2 + |\lambda|^2}$.

We shall use the quasi-linear hyper-radial potential to define the unperturbed Hamiltonian, which leads us to introduce the "hyper-spherical (H) string length" $l_{\rm H}$, or the "hyper-radius" R, as follows

$$R = \sqrt{\rho^2 + \lambda^2} = l_{\rm H}. \tag{14}$$

The two Jacobi vectors ρ, λ defined in Eqs. (3),(4), are shown in Fig. 3. Instead of two three-vectors ρ and λ, the hyper-spherical formalism introduces the hyper-radius R and the (new) hyper-angle χ by way of the "polar transformation"

$$\rho = R \sin\chi, \quad \lambda = R\cos\chi \quad \text{with} \quad 0 \le \chi \le \pi/2. \tag{15}$$

Thus we see that the expansion in Eq. (13) corresponds precisely to the expansion in powers of trigonometric functions of hyper-angles:

$$\begin{aligned} V_{\rm Y} &= \sigma\sqrt{\frac{3}{2}R^2(1+\sin 2\chi|\sin\theta|)}, \\ &= \sigma\sqrt{\frac{3}{2}}R\left(1+\frac{1}{2}\sin 2\chi|\sin\theta| + \cdots\right), \end{aligned} \tag{16}$$

the so-called hyper-spherical expansion, the first term being the "hyper-radial" term and the rest corresponding to higher-order "hyper-spherical harmonics".

The excited three-quark states then naturally fall into (non-degenerate) multiplets (labelled by the "grand angular momentum" quantum number K) of the broken O(6) symmetry, whose degeneracy is broken by the hyper-angular dependent perturbation, see Appendix A. These members/subsets of the (broken) O(6) multiplets coincide with the "ordinary" SU(6) multiplets of standard quark model in the two lowest lying bands (usually denoted by the harmonic oscillator label N=0,1), the only distinction being that in the N=2 band the first radial excitation of the N=0 ground state, a.k.a. the "Roper", does not fall into the K=2 band, but rather belongs to the K=0, hyper-radial excitation $\mathcal{N}_{\rm K} = 2$ "shell" consisting of exactly one state, see Appendix A 2. Thus, the symmetrized hyper-spherical harmonics are the most natural tool to describe the non-relativistic three-quark $\mathrm{SU}_{FS}(6)$ multiplets' wave functions and to study their mass splittings.

So, we shall solve the Schrödinger equation with the linear "hyper-radial" potential, see Appendix C 1, use its solutions as unperturbed states, and treat the second term in Eq. (16) as the (lowest-order) perturbation in "hyper-spherical harmonics", see Sects. IV A 1, V and Appendix B 1.

C. Non-perturbative approximations

The hyper-spherical formalism is often advertised as an economical way to tackle the general three-body problems in atomic and nuclear physics, but, in fact, it seems as if it had been tailor-made for the Y-string potential. We refer to specialized papers for technical aspects of this method (see for instance Refs. [24]). Here we recall just what is needed for our purposes; then, we use it below to tackle the problem non-perturbatively.

1. Y-string problem in hyper-spherical coordinates

We expand the three-quark wave function in hyper-spherical coordinates as

$$\Psi(R,\chi,\hat{\rho},\hat{\lambda}) \;=\; \sum_c \psi_c(R)\mathcal{Y}_c(\chi,\hat{\rho},\hat{\lambda}) \tag{17}$$

where $c = l_1, l_2, L, m$ and $K = l_1 + l_2 + 2m$ (occasionally one uses $[K]$ to denote the complete set of hyperspherical quantum numbers K, l_1, l_2, L, m) and

$$\mathcal{Y}_c(\chi,\hat{\rho}=\Omega_\rho,\hat{\lambda}=\Omega_\lambda) \;=\; u_{l_1,l_2,m}(\chi)[Y_{l_1}(\hat{\rho})\otimes Y_{l_2}(\hat{\lambda})]_L \tag{18}$$

$$u_{l_1,l_2,m}(\chi) \;=\; N_{l_1,l_2,m}F(-m,l_1+l_2+m+2|l_2+\frac{3}{2}|\sin^2\chi)(\cos\chi)^{l_1}(\sin\chi)^{l_2} \tag{19}$$

$$N_{l_1,l_2,m} \;=\; \frac{1}{\Gamma(l_2+3/2)}\sqrt{\frac{2(2m+l_1+l_2+2)\Gamma(m+l_1+l_2+2)\Gamma(m+l_2+3/2)}{m!\Gamma(m+l_1+3/2)}} \tag{20}$$

where $F(-n,n+a|c|z)$ is the Jacobi function

$$F(-n,n+a|c|z) \;=\; \sum_{s=0}^{n}(-1)^s\frac{n!\Gamma(c)\Gamma(a+n+s)}{(n-s)!s!\Gamma(a+n)\Gamma(c+s)}z^s \tag{21}$$

and $\Gamma(c)$ is Euler's Gamma function.

An important property of the hyper-spherical formalism is that a complicated three-body problem reduces to a set of (coupled) differential equations involving only the hyper-radial matrix elements of the potential $V(R)_{[K][K']} = \langle\Psi(\chi,\theta)_{[K']}|V(R,\chi,\theta)|\Psi(\chi,\theta)_{[K]}\rangle_{\rm hyp-ang.}$. The Schrödinger equation of the three-quark system then becomes a set of infinitely many coupled equations,

$$-\frac{1}{2\mu}\left[\frac{d^2}{dR^2}+\frac{5}{R}\frac{d}{dR}-\frac{K(K+4)}{R^2}+2\mu E\right]\psi_c(R)+\sqrt{\frac{3}{2}}\sigma R\sum_{c'}C_{c,c'}\psi_{c'}(R)=0 \tag{22}$$

with

$$\begin{aligned} C_{c',c} \;&=\; \langle\Psi(\chi,\theta)_{c'}|\sqrt{1+\sin 2\chi|\sin\theta_{12}|}|\Psi(\chi,\theta)_c\rangle_{\rm hyp-ang.} \\ &=\; \int_0^{\pi/2}\cos\chi^2\sin\chi^2 d\chi\int d\Omega_\rho d\Omega_\lambda\mathcal{Y}^*_{c'}\sqrt{1+\sin 2\chi|\sin\theta_{12}|}\mathcal{Y}_c \end{aligned} \tag{23}$$

where $\cos\theta_{12} = \hat{\rho}\cdot\hat{\lambda}$.

In special cases, such as the present one, some (finite) subset(s) of equations may decouple, due to the symmetries of the interaction potential. The spectrum of the system is then reduced to finding the eigenvalues of the following differential equation

$$\left[\frac{d^2}{dR^2}+\frac{5}{R}\frac{d}{dR}-\frac{K(K+4)}{R^2}-a_cR+k^2\right]\psi_c(R)=0 \tag{24}$$

where

$$a_c \;=\; 2\mu\sqrt{\frac{3}{2}}\sigma C_{c,c} \tag{25}$$

$$k^2 \;=\; 2\mu E \tag{26}$$

and the coupling matrix $C_{c'c}$

$$C_{c',c} \;=\; \sum_{L'}X_{c',c,L'}\int_0^{\pi/2}u^*_{c'}(\chi)u_c(\chi)\,\cos^2\chi\sin^2\chi d\chi\int_{-1}^{1}d\cos\theta_{12}P_{L'}(\cos\theta_{12})\sqrt{1+\sin 2\chi\,|\sin\theta_{12}|} \tag{27}$$

can be evaluated using the by now familiar angular re-coupling matrix $X_{c',c,L'}$,

$$\begin{aligned} X_{c',c,L'} \;&=\; \sqrt{(2l_{1'}+1)(2l_{2'}+1)(2l_1+1)(2l_2+1)}/2 \\ &\times\; (-1)^{L'+l_{1'}+l_{2'}}W(l_1,L',L,l_{2'};l_{1'},l_2)(l_{1'},0,l_1,0|L',0)(l_{2'},0,l_2,0|L',0) \end{aligned} \tag{28}$$

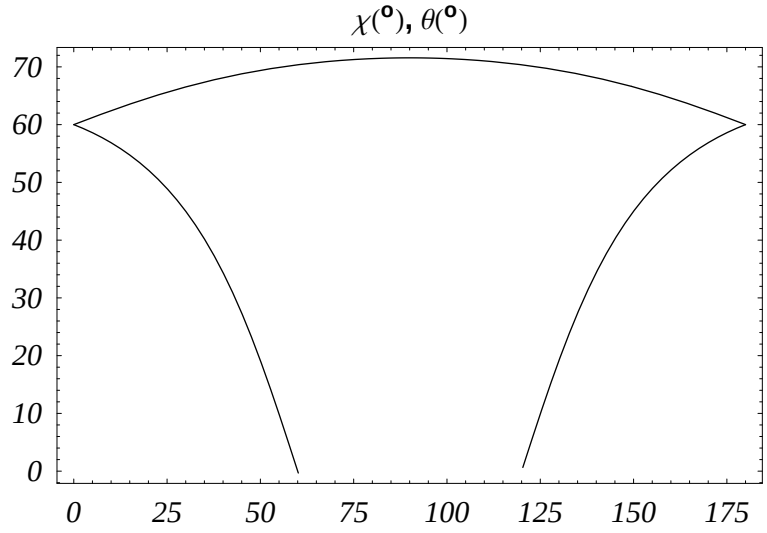

FIG. 4: The boundary in the χ vs. θ plane, between the regions in which the two- and the three-string potentials are appropriate, see Eqs. (30).

and the following two-dimensional integrals

$$\int_{-1}^{1} P_L(\cos\theta)\, d\cos\theta \int_0^{\frac{\pi}{2}} |u_c(\chi)|^2 \sqrt{1+|\sin\theta|\sin 2\chi}\,(\cos\chi\,\sin\chi)^2\, d\chi \tag{29}$$

We may numerically evaluate the necessary integrals, see Appendix B 2, so as to evaluate the hyper-angular matrix elements and recast the Schrödinger equation with the exact potential $V_{\rm Y}$ into a set of (coupled) hyper-radial equations for each "hyper-spherical harmonic" that can be solved numerically, see Appendix C.

2. The complete string potential in hyper-spherical coordinates

To finally solve the exact string problem, one must relax the Y-string potential approximation Eq. (5) and use the exact potential, i.e., all four of its incarnations, Eqs. (5,6a,6b,6c) depending on the angles formed by the three quarks. This means that one must first determine the boundary in the χ vs. θ plane between the regions in which the two- and the three-string potentials are appropriate, see Eqs. (30). There are three such boundaries, determined by the three inequalities, that merge continuously one into another at two "contact points", see Fig. 4.

$$\begin{aligned}
\cot\chi_1(\theta) &= \frac{-1}{\sqrt{3}\,\cos\theta+\sin\theta},\\
\cot\chi_2(\theta) &= \frac{1}{\sqrt{3}\,\cos\theta-\sin\theta},\\
\cot\chi_3(\theta) &= \frac{1}{3}\sqrt{5-2\cos^2\theta-2\,|\sin\theta|\,\sqrt{4-\cos^2\theta}}.
\end{aligned} \tag{30}$$

The hyper-angular matrix elements with these boundary conditions are evaluated in Appendix B 3. The rest of the calculation proceeds as in the case of the Y-string.

IV. RESULTS

A. The first-order perturbation theory

1. Y-string three-quark potential expanded to first powers of hyper-angles

In Appendix III C 2 we have calculated the matrix elements $\langle\, {\rm N_K, K}\,[SU_{\rm FS}(6), L^P]\,|\,\frac{2\,|\boldsymbol{\rho}\times\boldsymbol{\lambda}|}{R}\,|\,{\rm N_K, K}\,[SU_{\rm FS}(6), L^P]\,\rangle$ for the states/$SU_{\rm FS}(6)$ multiplets of the three lowest ${\rm N_K}(=0,1,2)$ bands: $|\,{\rm N_K}=0, {\rm K}=0, [56,0^+]\,\rangle$, $|\,{\rm N_K}=1, {\rm K}=$

$1,[70,1^-]\rangle$, and $|\,\mathrm{N_K}=1,\mathrm{K}=0,[56',0^+]\,\rangle$, $|\,\mathrm{N_K}=2,\mathrm{K}=2,[56,2^+]\,\rangle$, $|\,\mathrm{N_K}=2,\mathrm{K}=2,[70,0^+]\,\rangle$, $|\,\mathrm{N_K}=2,\mathrm{K}=2,[70,2^+]\,\rangle$, $|\,\mathrm{N_K}=2,\mathrm{K}=2,[20,1^+]\,\rangle$ where we separate each matrix element into its hyper-radial and hyper-angular parts:

$$\langle\,\psi_{\mathrm{N_K,[K]}}\,|\,\frac{2\,|\boldsymbol{\rho}\times\boldsymbol{\lambda}|}{R}\,|\,\psi_{\mathrm{N_K,[K]}}\,\rangle\;=\;\langle\,\psi_{\mathrm{N_K,K}}\,|\,R\,|\,\psi_{\mathrm{N_K,K}}\,\rangle_{\mathrm{hyp-rad}}\langle\,\psi_{\mathrm{[K]}}\,|\,\sin 2\chi\,\sin\theta_{\rho\lambda}\,|\,\psi_{\mathrm{[K]}}\,\rangle_{\mathrm{hyp-ang}},\tag{31}$$

where [K] denotes all the quantum numbers, such as the L, l_ρ, l_λ and their magnetic quantum numbers, associated with K. The hyper-radial matrix element can be calculated using the virial theorem, see Appendix C 2 , as follows:

$$\sigma\sqrt{\frac{3}{2}}\langle\,\psi_{\mathrm{N_K,K}}\,|\,R\,|\,\psi_{\mathrm{N_K,K}}\,\rangle_{\mathrm{hyp-rad}}\;=\;\frac{2}{3}\langle\,\psi_{\mathrm{N_K,K}}\,|\,H\,|\,\psi_{\mathrm{N_K,K}}\,\rangle_{\mathrm{hyp-rad}}=\frac{2}{3}E_{\mathrm{N_K,K}}.\tag{32}$$

The calculated energies of states with various values of K and L are listed in Table II. Note that the hyper-radial

TABLE II: The values of the unperturbed energy $E_{\mathrm{N_K,K}}$, the total energy $E_{\mathrm{K}}+\delta E_{\mathrm{K,L}}$ (in units of $\left(\sqrt{\frac{3}{2}}\frac{\sigma\,\hbar}{\sqrt{2\mu}}\right)^{\frac{2}{3}}$) and the three-body potential matrix elements' hyper-angular perturbation $\langle\frac{2\,|\boldsymbol{\rho}\times\boldsymbol{\lambda}|}{R^2}\rangle_{\mathrm{hyp-ang.}}=\langle\sin 2\chi\,|\,\sin\theta|\rangle_{\mathrm{hyp-ang.}}$ as well as some intermediate steps, for the various K = 0,1,2 states (with all allowed orbital waves L).

K	$\mathrm{N_K}$	$E^{(0)}_{\mathrm{N_K,K}}$	$[SU(6),L^P]$	$\frac{1}{3}\langle\sin 2\chi\,\vert\,\sin\theta\vert\rangle$	$E^{(0)}_{\mathrm{N_K,K}}+\delta E_{\mathrm{N_K,K,L}}$	$=E_{\mathrm{N_K,K,L}}$
0	0	3.8175	$[56,0^+]$	$\frac{2}{9}$	$\frac{11}{9}$ 3.8175	4.6658
1	0	4.6582	$[70,1^-]$	$\frac{2}{9}$	$\frac{11}{9}$ 4.6582	5.6934
0	1	5.2630	$[56,0^+]$	$\frac{2}{9}$	$\frac{11}{9}$ 5.2630	6.4326
2	0	5.4290	$[70,0^+]$	$\frac{8}{45}$	$\frac{53}{45}$ 5.4290	6.3942
2	0	5.4290	$[56,2^+]$	$\frac{44}{225}$	$\frac{269}{225}$ 5.4290	6.4907
2	0	5.4290	$[70,2^+]$	$\frac{52}{225}$	$\frac{277}{225}$ 5.4290	6.6837
2	0	5.4290	$[20,1^+]$	$\frac{4}{15}$	$\frac{19}{15}$ 5.4290	6.8767

matrix elements of the linear hyper-radial potential are identical for all K=2 band hyper-radial (ground) states. Therefore, all the energy differences among various K=2 multiplets are integer multiples of the energy splitting "unit" $\Delta_Y=\frac{2}{75}\sigma\sqrt{\frac{3}{2}}\langle R\rangle_{\mathrm{N,K}}=0.0327\sigma\langle R\rangle_{\mathrm{N,K}}=\frac{8}{225}E_{\mathrm{N_K,K}}$, which is just another manifestation of the validity of the Bowler-Tynemouth theorem [18] for this three-body potential solved in hyper-spherical coordinates.

This theorem has already been confirmed by J.M. Richard and P. Taxil [19], in the hyper-spherical formalism with linear two-body potentials. Our new contribution here is the (first) proof of this theorem, with the "area term" three-body potential as a perturbation, that holds even with non-harmonic oscillator unperturbed states. As in the case of Bowler and Tynemouth [18] the crucial ingredient of the proof is the permutation symmetry of the three-body potential. Other simple three body potentials that lack this permutation symmetry do not conform to this theorem.

Returning to the details of the spectrum, note, however, that the Roper $[56,0^+]$ multiplet (6.433) is roughly half-way between the $[70,0^+]$ (6.394) and the $[56,2^+]$ (6.491) multiplets, and for most practical intents and purposes degenerate with them both, as their mass difference is just one Δ_Y. The ratios of the K=2 and other K-values hyper-radial matrix elements need not be integers, or rational numbers any more, except in special cases, like in the harmonic oscillator, or in the Coulomb potential. In the Y-string potential the upward shift of the Roper $[56,0^+]$ ($5.263\rightarrow 6.433$) is proportional to its unperturbed energy (4.082), as is the upward shift of the K=1 odd-parity states ($4.658\rightarrow 5.693$), and the coefficient of proportionality $\frac{11}{9}$ is equal for these two states.

In other words, the Y-string potential does not move the Roper below the K=1 odd-parity resonances, at least not in perturbation theory. Note, however, that the largest correction in this hyper-spherical first-order perturbative approximation relative to the unperturbed value is 27%, which still does not justify a perturbative treatment. We shall therefore try and estimate the effects of the exact potential, i.e. the whole power series at once.

B. Non-perturbative results

1. *The three-string potential*

Instead of expanding $\sqrt{1+x}$, where $x=\sin\theta\sin 2\chi$, in a power series we may numerically integrate the double integral

$$\int_0^\pi P_L(\cos\theta)\,\sin\theta\,d\theta\int_0^{\frac{\pi}{2}}|u_{c,K,L}(\chi)|^2\,\sqrt{1+\sin\theta\sin\,2\chi}\,(\cos\chi\,\sin\chi)^2\,d\chi.\tag{33}$$

Such hyper-angular potential matrix elements are the coefficients multiplying the linear hyper-radial potential that appears in the (new) hyper-radial Schrödinger equation; that equation, in turn, can be solved exactly, i.e. without the use of the perturbation theory, and the resulting energy eigenvalues are listed in Table III. We can compare these exact results with the first-order perturbation theory: the energy eigenvalues, after rescaling, see Appendix C 1, are proportional to $E \sim \left(\sqrt{\frac{3}{2}}\frac{\sigma\,\hbar}{\sqrt{2\mu}}\langle\sqrt{1+\sin 2\chi|\sin\theta|}\rangle_{\rm hyp-ang.}\right)^{\frac{2}{3}}$ (modulo an overall hyper-radial dependent factor that is the same for all K=2 ground states). This can be (double) Taylor-expanded and yields $\left(\sqrt{\frac{3}{2}}\frac{\sigma\,\hbar}{\sqrt{2\mu}}\left(1+\frac{1}{2}\langle\sin 2\chi|\sin\theta|\rangle_{\rm hyp-ang.}\right)\right)^{\frac{2}{3}} \simeq \left(\sqrt{\frac{3}{2}}\frac{\sigma\,\hbar}{\sqrt{2\mu}}\right)^{\frac{2}{3}}\left(1+\frac{1}{3}\langle\sin 2\chi|\sin\theta|\rangle_{\rm hyp-ang.}\right)$, which is the first perturbative correction.

TABLE III: The values of the unperturbed energy $E^{(0)}_{\rm N_K,K}$, the total energy $E_{\rm K}$ (in units of $\left(\sqrt{\frac{3}{2}}\frac{\sigma\,\hbar}{\sqrt{2\mu}}\right)^{\frac{2}{3}}$) and the three-body potential matrix elements' hyper-angular exact non-perturbative matrix element $\langle V_{\rm Y}\rangle_{\rm hyp-ang.} = \langle\sqrt{1+\sin 2\chi|\sin\theta|}\rangle_{\rm hyp-ang.}$, for the various K = 0,1,2 states (with all allowed orbital waves L).

K	$\rm N_K$	$E^{(0)}_{\rm N_K,K}$	$[SU(6),L^P]$	$\langle\sqrt{1+\sin 2\chi\vert\sin\theta\vert}\rangle_{\rm hyp-ang.}$	$\langle V_{\rm Y}\rangle^{2/3}_{h.a.}E^{(0)}_{\rm K}$	$= E_{\rm N_K,K,L}$
0	0	3.8175	$[56,0^+]$	1.2876	1.18355×3.8175	4.5182
1	0	4.6582	$[70,1^-]$	1.2876	1.18355×4.6582	5.5132
0	1	5.2630	$[56,0^+]$	1.2876	1.18355×5.2630	6.2290
2	0	5.4290	$[70,0^+]$	1.2350	1.15110×5.4290	6.2493
2	0	5.4290	$[56,2^+]$	1.2560	1.16411×5.4290	6.3199
2	0	5.4290	$[70,2^+]$	1.2981	1.18998×5.4290	6.4604
2	0	5.4290	$[20,1^+]$	1.3402	1.21557×5.4290	6.5993

Thus we may expect the exact result to be smaller than the first-order perturbative result on two accounts: 1) from the inclusion of all orders in the expansion of the square-root in the potential: the even- and odd-order terms have opposite signs (this is an alternating series), see e.g. the first ten terms

$$\sqrt{1+x} \;=\; 1+\frac{x}{2}-\frac{x^2}{8}+\frac{x^3}{16}-\frac{5\,x^4}{128}+\frac{7\,x^5}{256}-\frac{21\,x^6}{1024}+\frac{33\,x^7}{2048}-\frac{429\,x^8}{32768}+\frac{715\,x^9}{65536}+\mathcal{O}(x^{10}), \tag{34}$$

with rapidly decaying coefficients, meaning that the most important correction to the linear $\frac{x}{2}$ term is the negative $\frac{x^2}{8}$ one; and 2) from the inclusion of all orders in the expansion of the two-thirds-root. Manifestly, both of these effects lead to the breaking of the "integer-value rule" of Bowler theorem for the energy splittings of the K=2 multiplets, but only at the 1% level.

Note the overall reduction of the potential matrix elements, as compared with the first-order perturbation theory results: the largest correction in the hyper-spherical non-perturbative approach relative to the unperturbed value is 22%, which is less than the 27% in the hyper-spherical non-perturbative approximation. This fact justifies *ex post facto* the perturbative treatment in Sect. IV A.

2. The complete string results

Finally, the results of the solution to the complete string potential problem are shown in Table IV, which shows a slight $\mathcal{O}(<1\%)$, yet clear increase, across the board, of the total energy over the Y-string approximation results in Table III.

The overall shifts of energies in this calculation would hardly justify the effort it took to complete it, were it not for their (relative) effects on the K=2 level splittings, which are substantial: The violations of the "Bowler-Tynemouth theorem" increase to 13% and 9% for the $[20,1^+]$-$[70,2^+]$, and $[70,2^+]$-$[56,2^+]$ mass differences, respectively, as compared with the $[56,2^+]$-$[70,0^+]$ one. Note, that the Roper multiplet $[56,0^+]$ (6.2340) has moved below the $[70,0^+]$ (6.2665) multiplet, but remains, for all practical purposes degenerate with it.

V. SUMMARY AND DISCUSSION

In summary, we have studied the low-lying states in the three quark spectra confined by a pure Y-string potential, i.e. without any two-quark potentials, in three different approximations, see Table V and Fig. 5.

TABLE IV: The values of the unperturbed energy $E^{(0)}_{\mathrm{N_K,K}}$, the total energy E_{K} (in units of $\left(\sqrt{\frac{3}{2}}\frac{\sigma\,\hbar}{\sqrt{2\mu}}\right)^{\frac{2}{3}}$) and the exact three-body potential hyper-angular non-perturbative matrix element $\langle V_{\mathrm{string}}\rangle_{\mathrm{hyp-ang.}}$, taken from Table X, as well as some intermediate steps, for the various K = 0,1,2 states (with all allowed orbital waves L).

K	$\mathrm{N_K}$	$E^{(0)}_{\mathrm{N_K,K}}$	$[SU(6), L^P]$	$\langle V_{\mathrm{string}}\rangle_{\mathrm{hyp-ang.}}$	$\langle V_{\mathrm{string}}\rangle^{2/3} E^{(0)}_{\mathrm{K}}$	$= E_{\mathrm{N_K,K,L}}$
0	0	3.8175	$[56, 0^+]$	1.2891	1.18449×3.8175	4.5218
1	0	4.6582	$[70, 1^-]$	1.2891	1.18449×4.6582	5.5176
0	1	5.2630	$[56, 0^+]$	1.2891	1.18449×5.2630	6.2340
2	0	5.4290	$[70, 0^+]$	1.2401	1.15943×5.4290	6.2665
2	0	5.4290	$[56, 2^+]$	1.2584	1.16886×5.4290	6.3279
2	0	5.4290	$[70, 2^+]$	1.2985	1.19022×5.4290	6.4617
2	0	5.4290	$[20, 1^+]$	1.3404	1.21567×5.4290	6.5999

An attractive Y-string potential splits the N=K=2 band states into degenerate $\mathrm{SU}_{FS}(6)$ multiplets: $[20, 1^+], [70, 2^+], [56, 2^+], [70, 0^+]$, ordered in descending mass, and following approximately the separation rule of 2:2:1. The best accuracy results lead to 2.25:2.18:1 splitting i.e. to a maximum violation of this rule is less than 13%.

The mass difference between the first (hyper-) radial excitation of the ground state, that is the "Roper multiplet" $[56', 0^+]$, and the odd-parity K=N=1 $[70, 1^-]$ multiplet is entirely determined by the difference between and the first (hyper-) angular and the first (hyper-) radial excitation eigen-energies in a linearly rising hyper-radial potential, which is always negative. In other words, the Roper resonance cannot be lowered below the odd-parity K=N=1 states in this potential, irrespective of the string tension constant and the quark masses, which are the only free parameters.

TABLE V: The eigen-energies (in units of $\left(\sqrt{\frac{3}{2}}\frac{\sigma\,\hbar}{\sqrt{2\mu}}\right)^{\frac{2}{3}}$) of the unperturbed solution to the hyper-central approximation, (see the text) $E^{(0)}_{\mathrm{N_K,K}}$, one perturbative ($E^{(1)}_{\mathrm{N_K,K,L}}$), and two non-perturbative approximations ($E^{(2)}_{\mathrm{N_K,K,L}}$, $E^{(3)}_{\mathrm{N_K,K,L}}$) to the Y-string potential, where the fourth one $E^{(Y)}_{\mathrm{K}}$ is the exact (numerical) result, for the various low-lying K = 0,1,2 states (with all allowed orbital waves L). The last column shows the results for the Δ-string potential of J.M. Richard and Taxil [19].

K	$\mathrm{N_K}$	$[SU(6), \mathrm{L^P}]$	$E^{(0)}_{\mathrm{N_K,K}}$	$E^{(1)}_{\mathrm{N_K,K,L}}$	$E^{(2)}_{\mathrm{N_K,K,L}}$	$E^{(Y)}_{\mathrm{N_K,K,L}}$	$E^{(\Delta)}_{\mathrm{N_K,K,L}}$
0	0	$[56, 0^+]$	3.8175	4.6658	4.5182	4.5218	5.3592
1	0	$[70, 1^-]$	4.6582	5.6934	5.5132	5.5176	6.5395
0	1	$[56, 0^+]$	5.2630	6.4326	6.2290	6.2340	7.3885
2	0	$[70, 0^+]$	5.4290	6.3942	6.2493	6.2665	7.5409
2	0	$[56, 2^+]$	5.4290	6.4907	6.3199	6.3279	7.5731
2	0	$[70, 2^+]$	5.4290	6.6837	6.4604	6.4617	7.6377
2	0	$[20, 1^+]$	5.4290	6.8767	6.5993	6.5999	7.7022

Next we compare our Y-string three-quark potential results with the Δ-shaped string ones, as obtained in Ref. [19]. A quick look at Table V shows that the Δ-string eigen-energies are substantially higher than the corresponding Y-string ones, with identical string tension σ.

In order to check the (re)scaling hypothesis *viz.* if the low-lying spectra are identical after a (re)scaling of the string tension $\sigma_Y \to f\sigma_\Delta$, we tabulate the rescaled Δ-string energies next to the exact Y-string ones in Table VI. The string tension scaling factor f is fixed at $\frac{5\,\pi}{32}\sqrt{\frac{3}{2}}1.2891 = 0.775001$, which ensures that the three lowest-lying states/bands, *viz.* $(K = 0, N_K = 0, 1)$ and $(K = 1, N_K = 0)$, have identical eigen-energies, see Table VI. Note that this value of f is substantially different from 0.5493, which was used in Ref. [12]. The energy rescaling factor is just the $\frac{2}{3}$ power of f: $f^{2/3} = \left(\frac{5\,\pi}{32}\sqrt{\frac{3}{2}}1.2891\right)^{\frac{2}{3}} = 0.843726$. One can see that the overall effect of the Y-string potential on the spectrum does not amount to a mere rescaling of the energy, i.e. of the string tension σ in the Δ-string potential: the Y-string potential pushes all of the K=2 states up in energy, some less others more, whereas the Δ-string shifts some K=2 states up, others down and leaves third states unchanged relative to the Y-string states. Moreover, the size of the energy splitting of the K=2 band states in the Y-string (0.381684) is more than two times bigger than in the rescaled Δ-string potential (0.155752).

Of course, the ordering of the shells/bands and of states within shells remains identical in both potentials, because it is controlled by the permutation symmetry of the states and of the potentials, as well as by the convexity of the

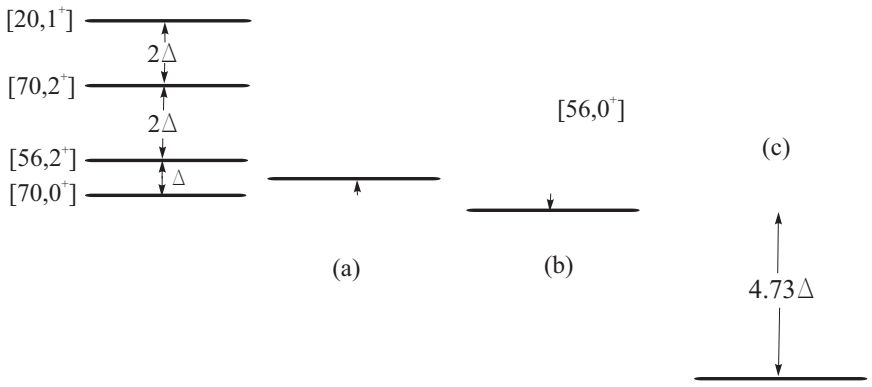

FIG. 5: Depiction of the energy splitting of the K = 2 states of the hyper-spherical linear potential spectrum due to attractive three-body potentials: (a) the first-order perturbation approximation to the first-power expansion of the "three-string" potential (see text for definition); (b) exact non-perturbative results of the "complete-string" potential and (c) the (rescaled) Δ-string potential results of J.M. Richard and Taxil [19]. The left-hand side of the diagram involving the $[20, 1^+]$, $[70, 2^+]$, $[56, 2^+]$, $[70, 0^+]$ multiplets is common to both kinds of potentials, follows the Bowler-Tynemouth rule 2:2:1 to 1%; only the position of the $[56, 0^+]$ multiplet (containing the Roper resonance) is variable.

TABLE VI: The eigen-energies (in units of $\left(\frac{\sigma\hbar}{\sqrt{2\mu}}\right)^{\frac{2}{3}}$) of the Y-string potential $E^{(Y)}_{\rm K}$ and the Δ-string potential results of J.M. Richard and Taxil [19] rescaled by factor $f^{2/3} = 0.843726$, see text, for the low-lying K = 0,1,2 states.

K	N_K	$[SU(6), L^P]$	$E^{(Y)}_{N_K,K,L}$	$f^{2/3}E^{(\Delta)}_{N_K,K,L}$
0	0	$[56, 0^+]$	5.1761	5.1761
1	0	$[70, 1^-]$	6.3160	6.3160
0	1	$[56, 0^+]$	7.1360	7.1360
2	0	$[70, 0^+]$	7.1733	7.2832
2	0	$[56, 2^+]$	7.2437	7.3143
2	0	$[70, 2^+]$	7.3968	7.3767
2	0	$[20, 1^+]$	7.5550	7.4390

potential in the hyper-radial coordinate. The only qualitative difference between the Δ- and Y-string potentials that we found is that the "Roper" lies lower, as compared with the lowest lying K=2 state, in the Δ-string than in the Y-string potential.

So we may conclude that, neglecting relativistic and (the very important) HFI effects, the three lowest-lying bands of states that form the (only) set of well-established resonances so far, do *not* allow a clear distinction to be made between these two types of potentials. This may, but need not, be a surprise, as these two string potentials have (very) different functional forms in configuration space, which we (naively) expected to predict different physics. But it turns out that the Bowler-Tynemouth theorem holds for general hyper-radial potentials, not just for the harmonic oscillator one, and then the separation of states in the K=2 shell is (tightly) constrained by the permutation symmetry.

The technical complexities of the Y-string potential, such as the orbital angular momentum dependence, which is responsible for some basic features of the spectrum, and the coupling of the radial and angular motions, encountered above are difficult to deal with accurately with methods that are not based on the hyper-spherical coordinates. The latter technique is not widely familiar to many practitioners in this field, however. The hyper-spherical formalism has one unfortunate disadvantage: the relativistic kinetic energy of three quarks is not a function of only the hyper-radius and as such the method does not (readily) extend to relativistic energies. For this reason we expect the present results to be relevant and applicable only to heavy-quark baryon spectroscopy, such as *ccc*.

That may explain the reason why this three-string potential has often been replaced in practical calculations by various two-body approximations, that are more easily handled by standard methods. Several two-body approximations to this potential have been devised and compared with the exact Y-string potential in various limits [12, 13, 22]. The results of the two most systematic papers so far [13, 22] are inconclusive, however, because the former did not calculate the spectrum and the latter is based on the hyper-radial approximation, which does not split the various SU(6) multiplets within the K=2 shell.

There is only one possible clue to the shape of the confining potential in the lower end of the baryon resonance spectrum, *viz.* the Roper resonances (abnormally low) mass, that perhaps could be used to draw conclusions about

the existence and/or preponderance of one kind of potential over the other. We have shown, however, that the Y-shaped string always leads to a Roper resonance that is heavier than the lowest-lying odd-parity resonance, just like the Δ-string.

This does not mean, however, that there need not be any differences between the spectra of the Y- and the Δ-string potentials, rather, it means that one must go to the higher lying bands of states, and in particular to higher orbital angular momentum states, in order to see these differences. It remains to be seen just how high is high enough.

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APPENDIX A: THE HYPER-SPHERICAL APPROACH

1. Brief review

The matrix elements/expectation values of the three-body potential V between the hyper-spherical harmonics $\mathcal{Y}_{K,L,m_i}(\Omega_5)$ and $\mathcal{Y}_{K',L',m_i'}(\Omega_5)$ are defined by

$$\begin{aligned}\langle V_{\rm Y}(R)\rangle_{[K][K']} &= \int \mathcal{Y}^*_{K,L,m_i}(\Omega_5)\, V(\boldsymbol{\rho},\boldsymbol{\lambda})\, \mathcal{Y}_{K',L',m_i'}(\Omega_5)\, d\Omega^{(5)} \\ &= \int_0^{\pi/2} \cos^2\chi\, \sin^2\chi\, d\chi \int_0^{\pi} \sin\theta\, d\theta\, \mathcal{Y}^*_{K,L,m_i}(\Omega_5)\, V(\boldsymbol{\rho},\boldsymbol{\lambda})\, \mathcal{Y}_{K',L',m_i'}(\Omega_5). \end{aligned} \tag{A1}$$

where

$$\begin{aligned}\cos\theta &= \cos\theta_\rho\, \cos\theta_\lambda + \cos(\varphi_\rho - \varphi_\lambda)\, \sin\theta_\rho\, \sin\theta_\lambda \\ &= -\left(\frac{4\,\pi}{\sqrt{3}}\right) [Y_1(\Omega_\rho) \otimes Y_1(\Omega_\lambda)]_{L=0} \end{aligned} \tag{A2}$$

The infinitesimal volume element $dV = R^5 dR d\Omega^{(5)}$ and the infinitesimal hyper-spherical solid angle $d\Omega^{(5)}$ are given by

$$dV = R^5 dR d\Omega^{(5)}, \tag{A3}$$

$$d\Omega^{(5)} = \cos^2\chi\, \sin^2\chi\, d\chi\, \sin\theta\, d\theta\, d\Omega^{(3)}, \tag{A4}$$

$$d\Omega^{(3)} = d\phi\, d\Omega^{(2)}, \tag{A5}$$

$$d\Omega^{(5)} = \cos^2\chi\, \sin^2\chi\, d\chi\, d\Omega^{(2)}_\rho\, d\Omega^{(2)}_\lambda, \tag{A6}$$

where $d\Omega^{(2)}$ is the usual three-dimensional space differential solid angle. Then one has

$$\int d\Omega^{(2)} = 4\pi, \quad \int d\Omega^{(3)} = 8\pi^2, \quad \int d\Omega^{(5)} = \pi^3. \tag{A7}$$

The ground state matrix element $\langle V_{\rm Y}\rangle_{00}(R)$ is often called the hyper-radial potential $V_{\rm Y}(R)$; the integrals in the hyper-radial Y-string potential $V_{\rm Y}(R)$ can be evaluated numerically as

$$\begin{aligned} V_{\rm Y}(R) = \langle V_{\rm Y}(R)\rangle_{00} &= \int \mathcal{Y}^*_{0,0,0}(\Omega_\lambda) V_{\rm Y}(\boldsymbol{\rho},\boldsymbol{\lambda})\, \mathcal{Y}_{0,0,0}(\Omega_\rho)\, d\Omega^{(5)} \\ &= \frac{1}{\pi^3}\int V_{\rm Y}(\boldsymbol{\rho},\boldsymbol{\lambda})\, d\Omega^{(5)}, \\ &= \frac{8}{\pi}\int_0^{\pi/2} \cos^2\chi\, \sin^2\chi\, d\chi \int_0^{\pi} V_{\rm Y}(R,\chi,\theta)\, \sin\theta\, d\theta \\ &= \sigma\sqrt{\frac{3}{2}}R\frac{8}{\pi}\int_0^{\pi} \sin\theta\, d\theta \int_0^{\frac{\pi}{2}} \sqrt{1+|\sin\theta|\, \sin 2\chi}\, (\cos\chi\, \sin\chi)^2\, d\chi \\ &\simeq 1.2876\,\sigma\sqrt{\frac{3}{2}}R. \end{aligned} \tag{A8}$$

where $|\sin\theta| = \sqrt{1-\cos^2\theta}$. One can easily write down the power expansion of the three-body potential $V_{\rm Y}$; thus one obtains an expansion of the potential in (integer) powers of ordinary spherical harmonics and of $\sin 2\chi$, which can be combined into (new) hyper-spherical harmonics $\mathcal{Y}_{K,L,m_i}$ that can be used in Eqs. (A1), (A8) by using the (hyper-spherical) Clebsch-Gordan coefficients, or simply by brute-force numerical integration. Note that these matrix elements are "almost always" diagonal, i.e., that this potential does not connect hyper-spherical harmonics with different K, but only of the same K and L, though possibly with different l_ρ and l_λ values, due to properties of the three-dimensional space spherical harmonics.

2. K=0,1,2 hyper-spherical harmonics

Manifestly, in order to be able to evaluate Eq. (A1) one must know the explicit form of the hyper-spherical harmonics. Simonov, Ref. [17], has explicitly written down the $K = 0, 1, 2$ states' wave functions (hyper-spherical harmonics) and examined their permutation symmetry properties, although the higher-K symmetrized hyper-spherical harmonics have remained a widely unexplored topic [24].

The symmetries of the string potential/hamiltonian are: parity, permutation/spatial exchange of quarks, rotation, therefore we see that only wave functions with the same $P = (-1)^{l_1+l_2}$, L, and symmetry M, S, A can mix with each other. Let P_{ij} be the ij-th particle permutation/spatial exchange operator. The permutation symmetry can be examined using the following properties of the (ρ,λ) set of vectors with mixed symmetry [25]

$$P_{12}\vec{\rho} \quad \to -\vec{\rho} \tag{A9a}$$

$$P_{12}\vec{\lambda} \quad \to \vec{\lambda} \tag{A9b}$$

$$P_{13}\vec{\rho} \quad \to \tfrac{1}{2}\vec{\rho} - \tfrac{\sqrt{3}}{2}\vec{\lambda} \tag{A9c}$$

$$P_{13}\vec{\lambda} \quad \to -\tfrac{\sqrt{3}}{2}\vec{\rho} - \tfrac{1}{2}\vec{\lambda} \tag{A9d}$$

that furnish the basis for the two-dimension irrep of S_3. Using the above relations, the second-order scalar, vector, and tensor can be constructed and used to obtain the $K = 0, 1, 2$ hyper-spherical harmonics, see Ref. [17]. Thus, it turns out that the S_3 permutation group symmetrized hyper-spherical harmonics correspond precisely to different $\mathrm{SU}(6)_{FS}$ symmetry multiplets (Young diagrams/tableaux) of the three-quark system. The symmetrized hyper-spherical harmonics have been systematically developed only up to K=2, see comments in Ref. [24]. We show them in Table VII.

APPENDIX B: THE HYPER-ANGULAR MATRIX ELEMENTS

1. First-order expansion of the Y-string potential

The following is, of course, just a schematic representation of the total matrix elements, as the complete K=2 band wave functions may have two components with different partial orbital momentum components (l_ρ, l_λ) and identical

TABLE VII: The hyper-spherical harmonics, for K = 0,1,2 values.

K	n_K	$[SU_{FS}(6), L^P]$	$(l_\rho \times l_\lambda)_L$	hyper-spherical harmonic
0	1	$[56, 0^+]$	$(0\times 0)_0$	$\mathcal{Y}_1 = \frac{4}{\sqrt{\pi}}[Y_0 \otimes Y_0]_0$
1	3	$[70, 1^-]$	$(1\times 0)_1$	$\mathcal{Y}_{2a} = 4\sqrt{\frac{2}{\pi}}\cos\chi[Y_1 \otimes Y_0]_1$
		$[70, 1^-]$	$(0\times 1)_1$	$\mathcal{Y}_{2b} = 4\sqrt{\frac{2}{\pi}}\sin\chi[Y_0 \otimes Y_1]_1$
2	1	$[70, 0^+]$	$(0\times 0)_0$	$\mathcal{Y}_{3a} = \frac{8}{\sqrt{\pi}}(\cos\chi^2 - \sin\chi^2)[Y_0 \otimes Y_0]_0$
		$[70, 0^+]$	$(1\times 1)_0$	$\mathcal{Y}_{3b} = \frac{16}{\sqrt{3\pi}}\cos\chi\sin\chi[Y_1 \otimes Y_1]_0$
2	3	$[20, 1^+]$	$(1\times 1)_1$	$\mathcal{Y}_4 = \frac{16}{\sqrt{3\pi}}\cos\chi\sin\chi[Y_1 \otimes Y_1]_1$
2	5	$[56, 2^+]$	$(2\times 0)_2$ $+(0\times 2)_2$	$\mathcal{Y}_5 = \frac{16}{\sqrt{5\pi}}\frac{1}{\sqrt{2}}\left[\cos^2\chi[Y_2 \otimes Y_0]_2 + \sin^2\chi[Y_0 \otimes Y_2]_2\right]$
2	5	$[70, 2^+]$	$(2\times 0)_2$ $-(0\times 2)_2$	$\mathcal{Y}_{6a} = \frac{16}{\sqrt{5\pi}}\frac{1}{\sqrt{2}}\left[\cos^2\chi[Y_2 \otimes Y_0]_2 - \sin^2\chi[Y_0 \otimes Y_2]_2\right]$
		$[70, 2^+]$	$(1\times 1)_2$	$\mathcal{Y}_{6b} = \frac{16}{\sqrt{3\pi}}\cos\chi\sin\chi[Y_1 \otimes Y_1]_2$

total orbital angular momentum L, such as the K=2 band multiplets $[70, 0^+]$ and $[70, 2^+]$ (see above):

$$\begin{aligned}\langle\, K(L_f)\,|\,\sin 2\chi\,|\sin\theta|\,|\,K(L_i)\,\rangle_{\rm hyp-ang} &= \langle\,\psi_{K(L_f)}(\chi)\,|\,\sin 2\chi\,|\,\psi_{K(L_i)}(\chi)\,\rangle_{\rm hyp}\langle\, Y_{L_f}\,||\sin\theta\,||\,Y_{L_i}\,\rangle_{\rm ang}\\ &= \sum_{l_1,l_2}\langle\,\psi_{K(L_f)}(\chi)\,|\,\sin 2\chi\,|\,\psi_{K(L_i)}(\chi)\,\rangle_{\rm hyp}\\ &\times\ \langle\langle(l_{1f}\otimes l_{2f})^{M_f}_{L_f}\,||\sin\theta||\,(l_{1i}\otimes l_{2i})^{M_i}_{L_i}\rangle_{\rm ang}. \qquad \text{(B1)}\end{aligned}$$

We may drop the indices for the initial and the final state total orbital angular momenta $L_f = L_i = L; M_f = M_i = M$, as the angular matrix elements are diagonal in those quantum numbers. We use the spatial wave functions available in Refs. [16, 17, 21] to calculate the hyper-angular matrix elements $\langle\,\psi_{\rm K}\,|\,\frac{2\rho\lambda}{R^2}\,|\,\psi_{\rm K}\,\rangle_{\rm hyp-ang} = \langle\,\psi_{\rm K}\,|\,\sin 2\chi\,|\,\psi_{\rm K}\,\rangle_{\rm hyp-ang}$ and take the angular matrix elements $\langle\, Y_L\,||\sin\theta||\,Y_L\,\rangle_{\rm ang}$ from Table I to calculate the total hyper-angular matrix element $\langle\,\psi_K\,||\,\sin 2\chi\,|\sin\theta|\,|\,\psi_K\,\rangle_{\rm hyp-ang}$. We list the radial-, angular- and total matrix elements' values, as well as the results of some intermediate steps, in Table VIII.

TABLE VIII: The values of the three-body potential hyper-angular matrix elements $\langle\frac{2\rho\lambda}{R^2}\rangle_{\rm hyp} = \langle\sin 2\chi\,\rangle_{\rm hyp}$ and $\langle\sin 2\chi\,|\sin\theta|\rangle_{\rm hyp-ang}$, for the K = 0,1,2 states (in all the partial S-, P- and D-waves). In the $[56, 2^+]$ and the $[70, 2^+]$ entries "diag." denotes diagonal terms of the type $(2\times 0)_2 \times (2\times 0)_2$ and $(0\times 2)_2 \times (0\times 2)_2$, whereas "off-diag." denotes off-diagonal terms $(2\times 0)_2 \times (0\times 2)_2$ and $(0\times 2)_2 \times (2\times 0)_2$. These two kinds of contribution must be separated because they have different angular matrix elements $\frac{4}{\pi}\langle|\sin\theta|\rangle$.

K	$[SU_{FS}(6), L^P]$	$(l_\rho \times l_\lambda)_L$	N	$\langle\frac{2\rho\lambda}{R^2}\rangle$	$\frac{3\pi}{8}\langle\sin 2\chi\,\rangle$	$\frac{4}{\pi}\langle|\sin\theta|\rangle$	$\langle\sin 2\chi\,|\sin\theta|\rangle$	=
0	$[56, 0^+]$	$(0\times 0)_0$	3	$\frac{8}{3\pi}$	1	1	$\frac{2}{3}$	0.6667
1	$[70, 1^-]$	$(1\times 0)_1$	4	$\frac{8}{3\pi}$	1	1	$\frac{2}{3}$	0.6667
0	$[56, 0^+]$	$(0\times 0)_0$	5	$\frac{8}{3\pi}$	1	1	$\frac{2}{3}$	0.6667
2	$[20, 1^+]$	$(1\times 1)_1$	5	$\frac{128}{45\pi}$	$\frac{16}{15}$	$\frac{9}{8}$	$\frac{4}{5}$	0.8000
2	$[70, 0^+]$	$(0\times 0)_0$	5	$\frac{32}{15\pi}$	$\frac{4}{5}$	1	$\frac{8}{15}$	0.5333
2	$[70, 0^+]$	$(1\times 1)_0$	5	$\frac{128}{45\pi}$	$\frac{16}{15}$	$\frac{3}{4}$	$\frac{8}{15}$	0.5333
2	$[56, 2^+]$	diag.	5	$\frac{64}{25\pi}$	$\frac{24}{25}$	1	$\frac{16}{25}$	0.6400
2	$[56, 2^+]$	off-diag.	5	$\frac{32}{75\pi}$	$\frac{4}{25}$	$-\frac{1}{8}$	$-\frac{4}{75}$	-0.0533
2	$[56, 2^+]$	$(2\times 0)_2$ $+(0\times 2)_2$	5				$\frac{44}{75}$	0.5867
2	$[70, 2^+]$	diag.	5	$\frac{64}{25\pi}$	$\frac{24}{25}$	1	$\frac{16}{25}$	0.6400
2	$[70, 2^+]$	off-diag.	5	$-\frac{32}{75\pi}$	$-\frac{4}{25}$	$-\frac{1}{8}$	$\frac{4}{75}$	0.0533
2	$[70, 2^+]$	$(2\times 0)_2$ $-(0\times 2)_2$	5				$\frac{52}{75}$	0.6933
2	$[70, 2^+]$	$(1\times 1)_2$	5	$\frac{128}{45\pi}$	$\frac{16}{15}$	$\frac{39}{40}$	$\frac{52}{75}$	0.6933

2. The Y-string potential

The total matrix elements may have two components with different partial orbital momenta (l_ρ, l_λ) and identical total orbital angular momentum L, e.g. the K=2 band multiplets $[70, 0^+]$ and $[70, 2^+]$: where

$$\langle\, \psi_{K(L_f)}(\chi)\,|\,\sqrt{1+\sin 2\chi|\sin\theta|}\,|\,\psi_{K(L_i)}(\chi)\,\rangle_{\rm hyp} \;=\; \int_0^{\frac{\pi}{2}} \left(\frac{1}{2}\sin 2\chi\right)^2 |\psi_{K,L}(\chi)|^2\,\sqrt{1+\sin 2\chi|\sin\theta|}\,d\chi. \qquad \text{(B2)}$$

and we are left with a double integral over the product of even-L order Legendre polynomials $P_L(x)$, the potential $\sqrt{1+\sin 2\chi|\sin\theta|}$ and the hyper-angular wave function squared $|\psi_{K,L}(\chi)|^2$:

$$\begin{aligned}\frac{1}{2}\int_{-1}^{1} d\cos\theta P_L(\cos\theta)\,\langle\,\psi_{K(L_f)}\,|\,\sqrt{1+\sin 2\chi|\sin\theta|}\,|\,\psi_{K(L_i)}\,\rangle \;&=\; \frac{1}{2}\int_{-1}^{1} d\cos\theta P_L(\cos\theta)\int_0^{\frac{\pi}{2}} d\chi\,\left(\frac{1}{2}\sin 2\chi\right)^2 \\ &\times\; |\psi_{K,L}(\chi)|^2\,\sqrt{1+\sin 2\chi|\sin\theta|}. \qquad \text{(B3)}\end{aligned}$$

We use the K=0,1,2 hyper-angular wave functions available in Appendix A 2 to calculate these hyper-angular matrix elements and take the angular momentum coefficients from Table I. The integral can be rewritten in terms of new variables $z = \cos 2\chi$ and $x = \cos\theta$ as follows

$$\begin{aligned}\frac{1}{2}\int_{-1}^{1} d\cos\theta P_L(\cos\theta)\,\langle\,\psi_{K(L_f)}(\chi)\,|\,\sqrt{1+\sin 2\chi|\sin\theta|}\,|\,\psi_{K(L_i)}(\chi)\,\rangle_{\rm hyp} \;&=\; \frac{1}{2}\int_{-1}^{1} dx P_L(x)\,\left(\frac{1}{2}\right)^3\int_{-1}^{1}\sqrt{1-z^2}dz \\ &\times\; |\psi_{K,L}(z)|^2\,\sqrt{1+\sqrt{1-z^2}\sqrt{1-x^2}}, \qquad \text{(B4)}\end{aligned}$$

and evaluated numerically; we list the results in Table IX.

TABLE IX: The values of the three-body potential hyper-angular matrix elements $\langle\sqrt{1+\sin 2\chi|\sin\theta|}\rangle_{\rm hyp-ang.}$ for the K = 0,1,2 states (in all the partial S-, P- and D-waves).

K	$[SU_{FS}(6), L^P]$	$\langle\sqrt{1+\sin 2\chi\|\sin\theta\|}\rangle_{\rm hyp-ang.}$
0	$[56, 0^+]$	1.2876
1	$[70, 1^-]$	1.2876
0	$[56, 0^+]$	1.2876
2	$[20, 1^+]$	1.3402
2	$[70, 0^+]$	1.2350
2	$[56, 2^+]$	1.2560
2	$[70, 2^+]$	1.2981

3. The complete string potential

The boundary Eqs. (30) can be recast in the new variables $z = \cos 2\chi$ and $x = \cos\theta$ of Fabre de la Ripelle and Lassaut, Ref. [22], that are particularly useful in the integration over the solid hyper-angle, see Fig. 6:

$$\begin{aligned} z_1(x) \;&=\; \frac{5\,x^2 - 8\,x^4 + \sqrt{3}\,x\,\sqrt{1-x^2}}{1 - 3\,x^2 + 8\,x^4 + \sqrt{3}\,x\,\sqrt{1-x^2}}, \\ z_2(x) \;&=\; \frac{-5\,x^2 + 8\,x^4 + \sqrt{3}\,x\,\sqrt{1-x^2}}{-1 + 3\,x^2 - 8\,x^4 + \sqrt{3}\,x\,\sqrt{1-x^2}}, \\ z_3(x) \;&=\; \frac{2 + x^2 + \sqrt{4 - 5\,x^2 + x^4}}{-7 + x^2 + \sqrt{4 - 5\,x^2 + x^4}}. \qquad \text{(B5)}\end{aligned}$$

Due to the symmetry of the integrand we may take twice the integral over x from 0 to 1; moreover the integral can

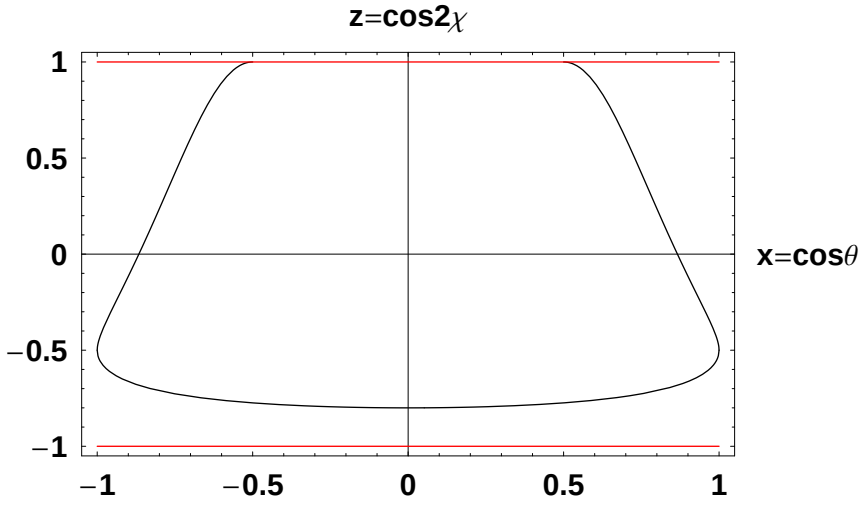

FIG. 6: The boundary in the $z = \cos 2\chi$ vs. $x = \cos\theta$ plane, between the regions in which the two- and the three-string potentials are appropriate, see Eqs. (B5).

be rewritten in terms of the new variables $z = \cos 2\chi$ and $x = \cos\theta$ as follows

$$\begin{aligned}
\frac{1}{2}\int_{-1}^{1} d\cos\theta P_L(\cos\theta)\,\langle\,\psi_{K(L_f)}(\chi)\,|\,V(\chi,\theta)\,|\,\psi_{K(L_i)}(\chi)\,\rangle_{\rm hyp} \;&=\; \left(\int_0^{\frac{1}{2}} dx\int_{z_3(x)}^{1} dz + \int_{\frac{1}{2}}^{1} dx\int_{z_3(x)}^{z_1(x)} dz\right)\\
\left(\frac{1}{2}\right)^3 P_L(x)\,\sqrt{1-z^2} \;&\times\; |\psi_{K(L)}(z)|^2\,\sqrt{\frac{3}{2}}\sqrt{1+\sqrt{1-z^2}\sqrt{1-x^2}}\\
&\times\; \left(\sqrt{1+\frac{z}{2}+\frac{x}{2}\sqrt{3(1-z^2)}}+\sqrt{1+\frac{z}{2}-\frac{x}{2}\sqrt{3(1-z^2)}}\right)\\
&+\; \left(\frac{1}{2}\right)^3\int_{\frac{1}{2}}^{1} P_L(x)\,dx\int_{z_1(x)}^{1}|\psi_{K(L)}(z)|^2\,\sqrt{1-z^2}dz\\
&\times\; \left(\sqrt{1-z}+\sqrt{1+\frac{z}{2}-\frac{x}{2}\sqrt{3(1-z^2)}}\right), \qquad \text{(B6)}
\end{aligned}$$

where $z_{1,3}(x)$ are given by Eqs. (B5) defining the boundary in the right-hand-side half of the x vs. z plane.

TABLE X: The values of the three-body potential hyper-angular matrix elements $\langle V_{\rm string}\rangle_{\rm hyp-ang.}$ for the K = 0,1,2 states (in all the partial S-, P- and D-waves).

K	$[SU_{FS}(6), L^P]$	$\langle V_{\rm string}\rangle_{\rm hyp-ang.}$
0	$[56, 0^+]$	1.289126
1	$[70, 1^-]$	1.289126
0	$[56, 0^+]$	1.289126
2	$[20, 1^+]$	1.340371
2	$[70, 0^+]$	1.240090
2	$[56, 2^+]$	1.258379
2	$[70, 2^+]$	1.298492

K	a_0	a_1	a_2
0	3.82	5.26	6.54
1	4.66	5.99	7.19
2	5.43	6.67	7.81

TABLE XI: Energy eigenvalues of the three-quark states in the linear hyper-radial potential for the ground- (a_0), the first- (a_1) and the second (a_2) radially excited states, in natural units (see text).

APPENDIX C: THE HYPER-RADIAL SCHRÖDINGER EQUATION

1. Solving the hyper-radial equation

The eigenvalues of the three quark states can be obtained by solving the eigenvalue problem of the following equation

$$\left[\frac{d^2}{dR^2}+\frac{5}{R}\frac{d}{dR}-\frac{K(K+4)}{R^2}+2\mu E-2\mu\sqrt{\frac{3}{2}}\sigma R\right]\psi_{[K]}=0 \tag{C1}$$

By changing scale of the hyper-radial coordinate one can write

$$\left[\frac{d^2}{dx^2}+\frac{5}{x}\frac{d}{dx}-\frac{K(K+4)}{x^2}+a-x\right]\psi_{[K]}=0 \tag{C2}$$

with

$$R = \alpha x \tag{C3}$$

$$\alpha = \left(2\mu\sqrt{\frac{3}{2}}\sigma\right)^{-1/3} \tag{C4}$$

$$a = 2\mu E\alpha^2 \tag{C5}$$

Regular solution can be written $\psi_{[K]}=x^K u_{[K]}(x)$ with $u_{[K]}(0)=const.$ The equation for $u_{[K]}$ can be written as

$$\left[\frac{d^2}{dx^2}+\frac{2K+5}{x}\frac{d}{dx}+a-x\right]u_{[K]}=0 \tag{C6}$$

The above equation is solved with the boundary condition

$$u(0) = \text{finite} \tag{C7}$$

$$u(\infty) \to 0 \tag{C8}$$

The obtained eigenvalues are tabulated in Table XI.

2. Evaluation of the potential's hyper-radial matrix element

We do not need the hyper-radial wave functions in order to complete the calculation of the radial matrix elements; rather, we use the virial theorem

$$V(k)=\frac{2}{k+2}E(k)$$

to determine the expectation value $\langle V(k=1)\rangle=\frac{2}{3}E(k=1)$, where k is the power of r in the potential $V(k)\sim r^k$, in terms of the energy eigenvalue $E(k=1)$ of the unperturbed hamiltonian.

[1] X. Artru, Nucl. Phys. **B 85**, 442 (1975).

[2] H. G. Dosch and V. Mueller, Nucl. Phys. **B 116**, 470 (1976).
[3] J. Carlson, J.B. Kogut, and V. R. Pandharipande, Phys. Rev. **D 27**, 233 (1983); *ibid.* **D 28**, 2807 (1983).
[4] S. Capstick and N. Isgur, Phys. Rev. **D 34**, 2809 (1986).
[5] W. H. Blask, U. Bohn, M. G. Huber, B. Ch. Metsch, and H. R. Petry, Z. Phys. **A 337**, 327 (1990).
[6] Y. Koma, E.-M. Ilgenfritz, T. Suzuki, and H. Toki, Phys. Rev. **D 64**, 014015 (2001).
[7] G. S. Bali, Phys. Rep. **343**, 1 (2001).
[8] T. T. Takahashi, H. Matsufuru, Y. Nemoto, and H. Suganuma, Phys. Rev. Lett. **86**, 18 (2001); Phys. Rev D **65**, 114509 (2002).
[9] C. Alexandrou, P. De Forcrand, A. Tsapalis, Phys. Rev. **D 65**, 054503,(2002).
[10] M. Caselle, G. Delfino, P. Grinza, O. Jahn and N. Magnoli, J. Stat. Mech. **0603**, P008 (2006) [arXiv:hep-th/0511168].
[11] Ph. de Forcrand and O. Jahn, Nucl. Phys. A **755**, 475 (2005) [arXiv:hep-ph/0502039].
[12] U. Löring, B. C. Metsch, H. R. Petry, Eur. Phys. J. **A 10**, 395 (2001).
[13] B. Silvestre-Brac, C. Semay, I. M. Narodetskii, and A.I. Veselov, Eur.Phys.J. **C32**, 385 (2003). I.M. Narodetskii and M.A. Trusov, hep-ph/0307131v1.
[14] M. Šuvakov and V. Dmitrašinović, in preparation (2008).
[15] D. Gromes and I.O. Stamatescu, Nucl.Phys.**B 112**, 213 (1976); Z. Phys. **C 3**, 43 (1979).
[16] N. Isgur and G. Karl, Phys. Rev. **D19**, 2653 (1979).
[17] Yu. A. Simonov, Sov. J. Nucl. Phys. **3**, 461 (1966) [Yad. Fiz. **3**, 630 (1966)].
[18] K.C. Bowler and B.F. Tynemouth, Phys. Rev. **D 27**, 662 (1983); K.C. Bowler, P.J. Corvi, A.J.G. Hey, P.D. Jarvis and R.C. King, Phys.Rev. **D 24**, 197 (1981).
[19] J.M. Richard and P. Taxil, Nucl.Phys. **B 329**, 310, (1990).
[20] D.A. Varshalovich, A.N. Moskalev and V.K. Khersonskii, "*Quantum theory of angular momentum"*, World Scientific, Singapore (1988).
[21] D. Faiman and A.W. Hendry, Phys. Rev. **173**, 1720 (1968).
[22] M. Fabre de la Ripelle and M. Lassaut, Few-Body Systems **23**, 75 (1997).
[23] *Iwanami Suugaku Koushiki III*, by Moriguchi, Udagawa and Hitotsumatsu. Iwanami, Tokyo (in Japanese). The English translation of *Iwanami sugaku jiten*, Nihon Sugakkai henshu, is Encyclopedic Dictionary of Mathematics. 2nd ed., (4 vols.), Cambridge, MA, MIT Press, (1993).
[24] R. Krivec, Few Body Systems **25**, 199 (1998).
[25] J.M. Richard, Phys. Rep. **212**, 1-76, (1992).
[26] "the Y-string and the Δ-string potentials are essentially indistinguishable"
[27] It would be ideal if these two kinds of string potentials were to predict (dramatically) different baryon spectra that can be easily differentiated by experiment, but that turns out not to be the case
[28] It has been claimed that this approximation is exact [22], however, in the case(s) of zero individual orbital angular momenta (l_ρ, l_λ), i.e. for the radial excitations of the S-wave ground state, and presumably a good one for low values of the orbital angular momenta. We shall show below that this claim is incorrect, however.
[29] but not the N=0,1 shells states.